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# Pointfree pseudocompactness revisited

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**Abstract**

We give several internal and external characterizations of pseudocompactness in frames which extend (and transcend) analogous characterizations in topological spaces. In the case of internal characterizations we do not make reference (explicitly or implicitly) to the reals.


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*Keywords:* Frame; Pseudocompact; Paracompact; Countably compact

## 1. Introduction

In the context of pointfree topology, the notion of pseudocompactness first appeared in 1991 in Baboolal and Banaschewski [1] where the authors state that their adopted definition (not making use of homomorphisms from the frame of opens of the reals) is an internal characterization which was established by C. Gilmour. Subsequent to that, Banaschewski and Pultr [7] gave some characterizations within the category of completely regular frames in 1993. Further characterizations within the class of completely regular frames were obtained by Marcus [12], Walters-Wayland [18] who showed, amongst other things, that a completely regular frame is pseudocompact if and only if it admits only precompact uniformities, and by Hlongwa [9] who compared pseudocompactness to other weaker forms of compactness; namely, feeble compactness and countable compactness.

The first extensive characterizations in arbitrary frames appeared in Banaschewski and Gilmour [4]. Our aim in this paper is to establish a number of internal and also external characterizations of pseudocompactness for general frames. We remark that some of these are pointfree extensions of analogous characterizations in topological spaces that were given by Stephenson [16].

What distinguishes our proofs in the case of internal characterizations from those of Stephenson is that, whereas he uses "external" artifacts (maps into the reals) in certain instances, all our proofs are "internal" in the sense that they use only things residing within the frames in question.

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We record our deep indebtedness to Dona Strauss for helpful discussions (with the second-named author) pertaining constructions of homomorphisms from the frame of opens of the reals.

## 2. Preliminaries

In this section we recall a few definitions that we shall need and refer to Johnstone [11] for general background on frames. For a more algebraic treatment of this engaging subject, see Pultr [14].

A *frame* is a complete lattice $L$ in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and all $S \subseteq L$. We denote the top element and the bottom element of a frame by $1_L$ and $0_L$ respectively; omitting the subscript if no confusion may arise. The frame of open subsets of a topological space $X$ will be denoted by $\mathfrak{O}X$.

A *cover* $C$ of a frame is a subset with $\bigvee C = 1$. A cover $C$ *refines* a cover $D$ if for each $c \in C$ there exists $d \in D$ such that $c \leqslant d$. A subset $S$ of a frame is *locally finite* if there is a cover $C$ such that each element of $C$ meets only finitely many elements of $S$. In this case we say $C$ *finitizes* $S$. A frame is *paracompact* (respectively *countably paracompact*) if each cover (respectively each countable cover) has a locally finite refinement. It is *compact* (respectively countably compact) if each cover (respectively each countable cover) has a finite subcover.

A frame $L$ is *regular* if, for each $a \in L$, $a = \bigvee \{x \in L \mid x \prec a\}$, where $x \prec a$ means that there exists $s \in L$ such that $x \wedge s = 0$ and $s \vee a = 1$. This is equivalent to $x^* \vee a = 1$ for the *pseudocomplement* $x^* = \bigvee \{w \in L \mid w \wedge x = 0\}$ of the element $x$. It is *completely regular* if, for each $a \in L$, $a = \bigvee \{x \in L \mid x \prec\prec a\}$ where $x \prec\prec a$ means that there is a scale $(c_q \mid q \in \mathbb{Q} \cap [0, 1])$ such that $x = c_0, a = c_1$ and $c_q \prec c_p$ whenever $q < p$. It is *normal* if whenever $a \vee b = 1$, then there are elements $u$ and $v$ such that $u \wedge v = 0$, $u \vee a = v \vee b = 1$. An element $x$ of a frame is *dense* if $x^* = 0$.

A *frame homomorphism* is a map between frames that preserves finite meets, including the top element, and arbitrary joins, including the bottom element. A frame homomorphism is *dense* if it maps only the bottom to the bottom. A *quotient* of a frame $L$ is a frame $M$ that admits an onto frame homomorphism $L \to M$.

A *cozero element* of a frame $L$ is an element of the form $\varphi(\mathbb{R} \setminus \{0\})$ for some frame homomorphism $\varphi : \mathfrak{O}\mathbb{R} \to L$. The set of all cozero elements of $L$ is called the *cozero part* of $L$ and is denoted by $\mathrm{Coz}(L)$. A useful characterization is that $a \in \mathrm{Coz}(L)$ if and only if $a = \bigvee a_n$ where $a_k \prec\prec a_{k+1}$ for each $k$. For further properties of the cozero part of a frame see [2], [4] or [5].

## 3. External characterizations

We start our study by recalling that for a frame $L$, a frame homomorphism $h : \mathfrak{O}\mathbb{R} \to L$ is said to be *bounded* if there exist $p, q \in \mathbb{R}$ such that $h(p, q) = 1_L$. The frame is then called *pseudocompact* in case all frame homomorphisms $\mathfrak{O}\mathbb{R} \to L$ are bounded. Quite clearly, every subframe of a pseudocompact frame is pseudocompact.

We shall frequently use the following result from [4].

**Proposition 3.1.** *The following are equivalent for any frame $L$:*

(1) *$L$ is pseudocompact.*
(2) *Any sequence $a_1 \prec\prec a_2 \prec\prec \cdots$ such that $\bigvee a_n = 1_L$ in $L$ terminates*; *that is, $a_k = 1_L$ for some $k$.*
(3) *The $\sigma$-frame $\mathrm{Coz}(L)$ is compact.*

Next we collect some properties of the kinds of frames we shall use in characterizing pseudocompact frames externally.

**Lemma 3.2.** *A normal paracompact frame is pseudocompact iff it is countably compact.*

**Proof.** Obviously any countably compact frame is pseudocompact. Conversely, let $\{a_1, a_2, \ldots\}$ be a countable cover of a normal paracompact frame $L$. By [8, the corollary on p. 97] there are elements $b_n$ such that $b_n \prec a_n$ for each $n$ and $\bigvee b_n = 1_L$. Now by normality we have that $b_n \prec\prec a_n$, and therefore there are cozero elements $c_n$ such that

$b_n \prec c_n \prec a_n$ for each $n$. Thus $\{c_n \mid n \in \mathbb{N}\}$ is a cover of the $\sigma$-frame $\mathrm{Coz}(L)$. By pseudocompactness there are finitely many $c_n$ that have join $1_L$, and so there are finitely many $a_n$ that cover $L$. □

In view of the fact that Boolean frames are normal and paracompact (the latter proved in [17, Proposition 6]), we immediately obtain the following fact.

**Corollary 3.3.** *A Boolean frame is pseudocompact iff it is countably compact.*

For a background on metrizable frames, which we refer to in the next result, we recommend Pultr [13].

**Lemma 3.4.** *A metrizable frame is compact iff it is countably compact.*

**Proof.** The one implication is trivial. Conversely, let $M$ be a metrizable frame which is countably compact. Then, as was shown by Sun [17], $M$ is paracompact. Being a regular frame that is paracompact and pseudocompact, $M$ is compact [7, Corollary 1]. □

We now give some external characterizations of pseudocompactness.

**Proposition 3.5.** *The following are equivalent for any frame $L$:*

(1) *$L$ is pseudocompact.*
(2) *If $h: M \to L$ is a one-to-one frame homomorphism, then $M$ is pseudocompact.*
(3) *If $h: M \to L$ is a one-to-one frame homomorphism with $M$ normal and paracompact, then $M$ is countably compact.*
(4) *For every metrizable frame $M$, if $h: M \to L$ is a one-to-one frame homomorphism, then $M$ is compact.*
(5) *For any composition $\mathcal{O}\mathbb{R} \to M \to L$ of frame homomorphisms where the first map is onto and the second one-to-one, $M$ is compact.*

**Proof.** (1) $\Rightarrow$ (2). Let $a_1 \prec\prec a_2 \prec\prec \cdots$ in $M$ with $\bigvee a_n = 1_M$. Then $h(a_1) \prec\prec h(a_2) \prec\prec \cdots$ in $L$, and $\bigvee h(a_n) = 1_L$. Since $L$ is pseudocompact, there exists an index $k$ such that $h(a_k) = 1_L$. Since $h$ is one-one, $a_k = 1_M$.

(2) $\Rightarrow$ (3). Follows from Lemma 3.2.

(3) $\Rightarrow$ (4). Metrizable frames are regular and paracompact, and therefore normal by Proposition 3.4 in [15]. So the result follows from Lemma 3.4.

(4) $\Rightarrow$ (5). This is so since $\mathfrak{O}\mathbb{R}$ is metrizable and quotients of metrizable frames are metrizable.

(5) $\Rightarrow$ (1). Let $f: \mathcal{O}\mathbb{R} \to L$ be a frame homomorphism. Consider the factorization $\mathcal{O}\mathbb{R} \to f[\mathcal{O}\mathbb{R}] \to L$ where the first map maps as $f$ (and is therefore onto) and the second map is the inclusion. The hypothesis says $f[\mathcal{O}\mathbb{R}]$ is compact. Now $\{f(-n, n) \mid n \in \mathbb{N}\}$ is a cover of $f[\mathcal{O}\mathbb{R}]$; so by compactness there exists $k \in \mathbb{N}$ such that $f(-k, k) = 1_L$. Thus $L$ is pseudocompact. □

## 4. Internal characterizations

A filter base $F$ in a frame $L$ is called *completely regular* if for each $x \in F$ there exists $y \in F$ such that $y \prec\prec x$. As in spaces we say a filter base $F$ *clusters* if $\bigvee\{x^* \mid x \in F\} \neq 1$. A cover $C$ is *co-completely regular* if for each $c \in C$ there exists $d \in C$ such that $c \prec\prec d$. We shall say a subset $S$ of $\mathrm{Coz}(L)$ is *locally finite in* $\mathrm{Coz}(L)$ if it is finitized by a cover of $\mathrm{Coz}(L)$; that is, if it is finitized by a countable cover of $L$ consisting of cozero elements.

**Proposition 4.1.** *The following are equivalent for any frame $L$:*

(1) *$L$ is pseudocompact.*
(2) *Every subset of* $\mathrm{Coz}(L)$ *which is locally finite in* $\mathrm{Coz}(L)$ *is finite.*
(3) *Every countable completely regular filter base in $L$ clusters.*
(4) *Every countable co-completely regular cover of $L$ has a finite subcover.*

**Proof.** (1) ⇒ (2). If not, there is a countably infinite set $B = \{b_1, b_2, \dots\} \subseteq \mathrm{Coz}(L)$ which is locally finite in Coz $L$ and consisting of nonzero elements. Let $C$ be a cover of $\mathrm{Coz}(L)$ that finitizes $B$. Now define elements $a_n$, for $n \in \mathbb{N}$, as follows:

$$a_n = \bigvee \{x \in C \mid x \wedge b_k = 0 \text{ for all } k \geqslant n\}.$$

Then we clearly have that $a_n \leqslant a_{n+1}$ for each $n$, and each $a_n$ is in $\mathrm{Coz}(L)$ because it is a join of countably many cozero elements. Furthermore, if $c \in C$ then $c \leqslant a_k$ for some $k$ since $c$ meets only finitely many elements of $B$. Thus $A = \{a_n \mid n \in \mathbb{N}\}$ is a cover of $\mathrm{Coz}(L)$. Since $\mathrm{Coz}(L)$ is compact, $A$ has a finite subcover. This implies that $a_k = 1_L$ for some $k$; whence $b_k = 0$ since $b_k = b_k \wedge \bigvee \{x \in C \mid x \wedge b_i = 0 \text{ for all } i \geqslant k\} \leqslant \bigvee \{b_k \wedge x \mid x \wedge b_k = 0\}$. This contradiction proves the result.

(2) ⇒ (3). Let $F = \{x_1, x_2, \dots\}$ be a completely regular filter base in $L$. For each $n$ let $y_n = x_1 \wedge \dots \wedge x_n$ and note that $y_n \neq 0$ since $F$ is a filter base, and $y_n \geqslant y_{n+1}$. Now let $q \in \mathbb{N}$. Find $x_{n_1}, \dots, x_{n_q}$ in $F$ such that $x_{n_1} \prec\prec x_1, \dots, x_{n_q} \prec\prec x_q$. Then $x_{n_1} \wedge \dots \wedge x_{n_q} \prec\prec x_1 \wedge \dots \wedge x_q = y_q$. If we let $m = \max\{n_1, \dots, n_q\}$, then $y_m = x_1 \wedge \dots \wedge x_m \leqslant x_{n_1} \wedge \dots \wedge x_{n_q} \prec\prec y_q$; so that $y_m \prec\prec y_q$. We can therefore extract a subsequence $(y_{m_k})_{k \in \mathbb{N}}$ from $(y_n)$ such that $y_{m_k} \leqslant y_k$ for each $k$ and

$$\dots \prec\prec y_{m_2} \prec\prec y_{m_1} \prec\prec y_1.$$

Thus there are cozero elements $c_1, c_2, \dots$ and cozero elements $d_1, d_2, \dots$ such that

$$\dots \prec y_{m_k} \prec c_k \prec y_{m_{k-1}} \prec \dots \prec y_{m_2} \prec c_2 \prec y_{m_1} \prec c_1 \prec y_1$$

and

$$y_1^* \prec d_1 \prec y_{m_1}^* \prec d_2 \prec y_{m_2}^* \prec \dots \prec y_{m_{k-1}}^* \prec d_k \prec y_{m_k}^* \prec \cdots.$$

Since $y_n \leqslant x_n$ for each $n$, we have $x_n^* \leqslant y_n^*$ for each $n$. Hence if we can show that $\bigvee y_n^* \neq 1_L$ we shall be done. Suppose, by way of contradiction, that $\bigvee y_n^* = 1_L$. Then $D = \{d_n \mid n \in \mathbb{N}\}$ is a cover of $\mathrm{Coz}(L)$. We claim that it finitizes $C = \{c_n \mid n \in \mathbb{N}\}$. Given any $k \in \mathbb{N}$ choose $n(k)$ and $l(k)$ in $\mathbb{N}$ such that $d_k \prec y_{n(k)}^*$ and $c_{l(k)} \prec y_{n(k)}$. Since the sequence $(c_n)$ decreases and $y_{n(k)}^* \wedge y_{n(k)} = 0$, it follows that $d_k$ has nonzero meet with at most $c_1, \dots, c_{l(k)-1}$. This shows that the set $C$ is locally finite in $\mathrm{Coz}(L)$ and is therefore finite by the hypothesis. Say $C = \{c_{p_1}, \dots, c_{p_s}\}$ with $c_{p_1} \geqslant \dots \geqslant c_{p_s}$. Now

$$\bigvee_{i \in \mathbb{N}} c_i^* \geqslant \bigvee_{i \in \mathbb{N}} y_{m_i}^* = \bigvee_{n \in \mathbb{N}} y_n^* = 1_L.$$

But $\bigvee c_i^* = c_{p_s}^*$; so $c_{p_s}^* = 1_L$, which implies that $c_{p_s} = 0$ and hence $y_k = 0$ for some $k$. This is a contradiction.

(3) ⇒ (4). If not, there is a countable co-completely regular cover $C$ which has no finite subcover. Let $G = \{\bigwedge x^*(x \in F) \mid F \text{ is a finite subset of } C\}$. We claim that $G$ is a filter base. Let $c_1, \dots, c_m$ be finitely many elements of $C$. We need only show that $c_1^* \wedge \dots \wedge c_m^* \neq 0$. If this meet were 0, then for $d_1, \dots, d_m$ in $C$ with $c_i \prec\prec d_i$ we would have $c_1 \vee \dots \vee c_m \prec\prec d_1 \vee \dots \vee d_m$, whence $(c_1 \vee \dots \vee c_m)^* \vee (d_1 \vee \dots \vee d_m) = 1_L$ and therefore $d_1 \vee \dots \vee d_m = 1_L$ since $(c_1 \vee \dots \vee c_m)^* = c_1^* \wedge \dots \wedge c_m^* = 0$. But then this would mean $C$ has a finite subcover. Now let $g = x_1^* \wedge \dots \wedge x_m^*$ be an arbitrary element of $G$. Pick $y_i \in C$ such that $x_i \prec\prec y_i$ for each $i$. Then $y_i^* \prec\prec x_i^*$ for each $i$. Thus $y_1^* \wedge \dots \wedge y_m^*$ is an element of $G$ which is completely below $g$. Therefore $G$ is a completely regular filter base which is countable. But now we have a contradiction since, for each $c \in C$, $c^*$ is in $G$ and hence

$$\bigvee_{x \in G} x^* \geqslant \bigvee_{c \in C} c^{**} = 1_L,$$

contradicting the hypothesis.

(4) ⇒ (1). Let $(a_n)$ be a sequence such that $a_1 \prec\prec a_2 \prec\prec \cdots$ and $\bigvee a_n = 1_L$. Then $\{a_n \mid n \in \mathbb{N}\}$ is a countable co-completely regular cover. So there exist integers $n_1 < \dots < n_k$ such that $a_{n_1} \vee \dots \vee a_{n_k} = 1_L$. This implies that $a_{n_k} = 1_L$. □

Recall that a frame is *almost compact* if every cover has a finite subset the join of which is dense. Hong [10] has shown that a frame is almost compact if and only if every filter in it clusters. Now if a filter base does not cluster, then the filter it generates also does not cluster. From the characterization above we therefore have the following result.

**Corollary 4.2.** *Every almost compact frame is pseudocompact.*

An element $x$ of a frame is called *regular* in case $x = x^{**}$. Note that regular elements are precisely those that are pseudocomplements. The *semi-regularization* of a frame $L$ is the subframe $L_s$ generated by the regular elements of $L$.

**Corollary 4.3.** *A frame is pseudocompact iff its semi-regularization is pseudocompact.*

**Proof.** Let $L$ be a frame with a pseudocompact semi-regularization, and let $F$ be a countable completely regular filter base in $L$. The set $G = \{x^{**} \mid x \in F\}$ is easily checked to be a filter base in $L_s$ in light of the identity $a^{**} \wedge b^{**} = (a \wedge b)^{**}$. Now if $x \prec y$ in $L$ and $t$ is an element of $L$ witnessing this fact, then $x^{**} \wedge t^{**} = 0$ and $t^{**} \vee y^{**} = 1$; whence $x^{**} \prec y^{**}$ in $L_s$. Thus, for any $z$ and $w$ in $L$, if $z \prec\prec w$ then $z^{**} \prec\prec w^{**}$ in $L_s$. So $G$ is a countable completely regular filter base in $L_s$. Denote by $(\ )^{\circledast}$ the pseudocomplementation in $L_s$. Therefore $\bigvee\{g^{\circledast} \mid g \in G\} \neq 1_{L_s} = 1_L$. Now, for any $a \in L_s$, we clearly have $a^{\circledast} \leqslant a^*$. On the other hand, in view of the fact that $x \wedge y = 0$ if and only if $x^{**} \wedge y = 0$, we have that $a^* \leqslant \bigvee\{x^{**} \mid x \in L \text{ and } x \wedge a = 0\} \leqslant a^{\circledast}$. Consequently, $\bigvee\{x^* \mid x \in F\} = \bigvee\{g^* \mid g \in G\}$ because of the identity $u^* = u^{***}$. Thus $L$ is pseudocompact. $\square$

**Remark.** We have chosen to establish the foregoing result in the rather longwinded manner we did because, being an internal characterization, we wanted to prove it without reference to the reals. In fact this result follows from the following observation: If $L$ is a frame and $M$ a subframe of $L$ such that $\mathrm{Im}(h) \subseteq M$ for each frame homomorphism $h : \mathfrak{O}\mathbb{R} \to L$, then $L$ is pseudocompact if and only if $M$ is. Indeed, given any frame homomorphism $f : \mathfrak{O}\mathbb{R} \to L$, let $\tilde{f} : \mathfrak{O}\mathbb{R} \to M$ map as $f$. Then the boundedness of $\tilde{f}$ clearly implies that of $f$. Now if we let $L_c$ be the subframe of $L$ generated by $\mathrm{Coz}(L)$, then we have $L_c \subseteq L_s$ because every cozero element is a join of regular elements as $x \prec\prec a$ implies $x^{**} \leqslant a$. Furthermore, the image of any frame homomorphism $\mathfrak{O}\mathbb{R} \to L$ is contained in $L_c$ since all elements of $\mathfrak{O}\mathbb{R}$ are cozero and frame homomorphisms preserve cozero elements. Consequently we have that *a frame $L$ is pseudocompact iff $L_s$ is pseudocompact iff $L_c$ is pseudocompact.*

Banaschewski [3] calls a frame $M$ a *singly-generated extension* of a frame $L$ if $L$ is a subframe of $M$ and there is an element $c \in M$ such that $L \cup \{c\}$ generates $M$. Such an element is called a *generator* of $M$ *over* $L$. He goes on to show that in such a case every element of $M$ is expressible as $x = x_1 \vee (x_2 \wedge c)$ with $x_1 \leqslant x_2$ in $L$. Furthermore, if $c$ is dense, then the pseudocomplement of $x$ in $M$ coincides with the pseudocomplement of $x_2$ in $L$. Therefore, if $x \prec y$ in $M$ then $x^* \vee y = 1_M$ implies that $x_2^{\#} \vee y_2 = 1_L$ since $y \leqslant y_2$, where $(\ )^{\#}$ denotes pseudocomplementation in $L$. Consequently, if $x \prec\prec y$ in $M$ then $x_2 \prec\prec y_2$ in $L$.

Now seeing that “extension” in frames has subsequently been used differently (namely, $M$ is an extension of $L$ if there is a dense onto frame homomorphism $M \to L$), we prefer to say $M$ is a singly-generated *expansion* of $L$ if the above holds. This latter nomenclature of course comes from topology.

**Corollary 4.4.** *Let $M$ be a singly-generated expansion of $L$ with a dense generator. Then $M$ is pseudocompact iff $L$ is pseudocompact.*

**Proof.** Only one implication needs to be proved. So let $L$ be pseudocompact and $c$ be a dense generator of $M$ over $L$. Let $C$ be a countable co-completely regular cover of $M$. Using the notation of the discussion above, for each $x \in C$ choose any $x_2$ and put $\tilde{C} = \{x_2 \mid x \in C\}$. Since $x \leqslant x_2$ for each $x$, we have that $\tilde{C}$ is countable cover of $L$ which is co-completely regular in view of what we observed above and the fact that $C$ is co-completely regular. Therefore $\tilde{C}$ has a finite subcover, say, $D$. Now for each $d \in D$ find $y_d$ in $C$ such that, for some $u \leqslant d$ in $L$, $u \vee (d \wedge c) \prec\prec y_d$. This is possible since $C$ is co-completely regular. But now this implies that $(u \vee (d \wedge c))^* \vee y_d = 1_M$; whence $d^{\#} \vee y_d = 1_M$ by the Banaschewski result mentioned above. Thus $d \leqslant y_d$; and consequently $\{y_d \mid d \in D\}$ is a finite subcover extracted from $C$. $\square$

## 5. Other properties

In their extension of the concept of $C$-embedded subspaces to pointfree topology, Ball and Walters-Wayland [2] say an onto frame homomorphism $h : L \to M$ is a *$C$-quotient map* in case for every frame homomorphism $g : \mathfrak{O}\mathbb{R} \to M$, there is a frame homomorphism $f : \mathfrak{O}\mathbb{R} \to L$ such that $h \circ f = g$.

**Proposition 5.1.** *If $L$ is a pseudocompact frame then $L$ admits no $C$-quotient map $L \to \mathfrak{O}\mathbb{N}$.*

**Proof.** If not, let $h: L \to \mathfrak{O}\mathbb{N}$ be a $C$-quotient map. For each $n \in \mathbb{N}$ let $b_n = \{1, \ldots, n\}$. Then each $b_n$ is a cozero element of $\mathfrak{O}\mathbb{N}$ since $\mathfrak{O}\mathbb{N}$ is Boolean. Furthermore

$$b_1 \prec\prec b_2 \prec\prec \cdots \quad \text{and} \quad \bigvee b_n = 1_{\mathfrak{O}\mathbb{N}}.$$

Thus, in the language of [2], $B = \{b_n \mid n \in \mathbb{N}\}$ is a cozero tower in $\mathfrak{O}\mathbb{N}$. So by Theorem 7.2.7(6) in [2] there is a cozero tower $S$ in $L$ such that $h[S]$ refines $B$. Since $L$ is pseudocompact and $S$ is an increasing sequence of cozero elements each completely below the next, there exists $s \in S$ such that $s = 1_L$. So $1_{\mathfrak{O}\mathbb{N}} = h(s) \leqslant b_m$ for some $m \in \mathbb{N}$. This implies that $b_m = 1_{\mathfrak{O}\mathbb{N}}$; which is false. □

**Proposition 5.2.** *Let $L$ be a frame with the property that for all $a, b \in \mathrm{Coz}(L)$ with $a \vee b = 1_L$, $\uparrow a$ or $\uparrow b$ is compact. Then $L$ is pseudocompact.*

**Proof.** Suppose, by way of contradiction, that there is an unbounded frame homomorphism $h: \mathfrak{O}\mathbb{R} \to L$. For any $t \in \mathbb{R}$, if $h(x, t) = 0_L$ for all $x < t$ and $h(t, z) = 0_L$ for all $z > t$, then $h(t, \infty) = h(-\infty, t) = 0_L$; whence $h(-\infty, t + 1) = h(t - 1, \infty) = 1_L$ since $1_L = h(\mathbb{R})$ and $\mathbb{R} = (-\infty, t + 1) \cup (t, \infty)$, and similarly for the other case. So we may assume that for each $t \in \mathbb{R}$ there exists $s > t$ such that $h(t, s) \neq 0_L$. A similar argument holds if we assume that for each $t \in \mathbb{R}$ there exists $s < t$ such that $h(s, t) \neq 0_L$. Now fix $q \in \mathbb{R}$ and choose an increasing sequence $(t_n)_{n\in\mathbb{N}}$ inductively in $\mathbb{R}$ as follows: $t_1 = q$, and, with $t_n$ having been chosen, choose $t_{n+1}$ such that $h(t_n + 1, t_{n+1}) \neq 0_L$. Next define elements $a$ and $b$ in $L$ by

$$a = h\left((-\infty, q + 1) \cup \bigcup_{n \text{ even}} (t_n, t_{n+1} + 1)\right) \quad \text{and} \quad b = h\left(\bigcup_{n \text{ odd}} (t_n, t_{n+1} + 1)\right).$$

Then $a$ and $b$ are cozero elements of $L$ satisfying $a \vee b = 1_L$. We show that neither $\uparrow a$ nor $\uparrow b$ is compact. To see that $\uparrow a$ is not compact, consider the set $C = \{a \vee h(-\infty, t_n) \mid n \in \mathbb{N}\}$. Clearly $C$ is a cover of the frame $\uparrow a$. However, for any $n \in \mathbb{N}$, $a \vee \bigvee_{m\leqslant n} h(-\infty, t_m) \neq 1_L$ because for any odd integer $r > n$ we have

$$h(t_r + 1, t_{r+1}) \wedge \left(a \vee h\left(\bigcup_{m\leqslant n} l(-\infty, t_m)\right)\right) = 0_L$$

whilst $h(t_r + 1, t_{r+1}) \neq 0_L$. Similarly, $\uparrow b$ is not compact; and we thus have a contradiction. □

In [6] Banaschewski and Gilmour say a dense onto frame homomorphism $h: M \to L$ is a *one-point extension* of $L$ if there is a maximal element $s \in M$ such that $h$ induces an isomorphism $\downarrow s \to L$. They then prove that if $\bigvee : R\mathfrak{J}(\mathrm{Coz}(L)) \to L$ is a one-point extension of $L$, where $R\mathfrak{J}(\mathrm{Coz}(L))$ is the frame of regular ideals of $\mathrm{Coz}(L)$, then $L$ satisfies the hypothesis in the preceding proposition. We consequently have:

**Corollary 5.3.** *If the Stone–Čech compactification of a completely regular frame is a one-point extension, then the frame is pseudocompact.*

Hlongwa [9] has shown that if $x$ is a regular element of a pseudocompact completely regular frame, then $\uparrow x^*$ is pseudocompact. In a general case we have the following result.

**Proposition 5.4.** *Let $L$ be a pseudocompact frame and $a \in L$ be such that $\uparrow(a \vee a^*)$ is pseudocompact. Then $\uparrow a^*$ is pseudocompact.*

**Proof.** Let $h: \mathfrak{O}\mathbb{R} \to \uparrow a^*$ be a frame homomorphism and let $g: \uparrow a^* \to \uparrow(a \vee a^*)$ be the frame homomorphism $x \rightsquigarrow x \vee a$. Because $\uparrow(a \vee a^*)$ is pseudocompact, the composite $g \circ h: \mathfrak{O}\mathbb{R} \to \uparrow(a \vee a^*)$ is bounded and therefore there exists $r \in \mathbb{R}$ such that $1_L = g(h(-r, r)) = h(-r, r) \vee a$. Now define a map $f: \mathfrak{O}\mathbb{R} \to L$ by

$$f(U) = h(r, \infty) \wedge h(U) \wedge a \quad \text{if } r \notin U, \quad \text{and} \quad f(U) = h(-r, r) \vee h(U) \quad \text{otherwise}.$$

One checks easily that $f$ is a frame homomorphism. Since $L$ is pseudocompact there exists $s > r$ in $\mathbb{R}$ such that $f(-s, s) = 1_L$. Since $r \in (-s, s)$, we have that $1_L = f(-s, s) = h(-r, r) \vee h(-s, s) = h(-s, s)$. Therefore $h$ is bounded as required. □

We end by proving the "countable" version of the Banaschewski–Pultr [7] result that we cited in the proof of Lemma 3.3; namely, a completely regular frame is compact if and only if it is paracompact and pseudocompact. Hlongwa [9] has shown that a completely regular frame is pseudocompact if and only if every countable cover has a finite subset with a dense join. The latter is a pointfree enunciation of a property of topological spaces known to be equivalent to feeble compactness, which in turn is equivalent to pseudocompactness in the category of completely regular spaces.

**Proposition 5.5.** *A completely regular frame is countably compact iff it is pseudocompact and countably paracompact.*

**Proof.** The forward implication is immediate. Conversely, let $L$ be pseudocompact and countably paracompact and $A = \{a_n \mid n \in \mathbb{N}\}$ be a countable cover. For each $n$ let $b_n = a_1 \vee \cdots \vee a_n$, and note that the $b_n$ form an increasing cover of $L$. By countable paracompactness there is a cover $\{c_n \mid n \in \mathbb{N}\}$ such that $c_n \prec b_n$ for each $n$ [8, Proposition 7]. In light of the fact that if $x \prec u$ and $y \prec v$ then $x \vee y \prec u \vee v$, we may assume that the $c_n$ increase because the $b_n$ increase. Now pick $k$ such that $(c_1 \vee \cdots \vee c_k)^* = 0$. This implies that $c_k^* = 0$, and therefore $b_k = 1$. Thus $A$ has a finite subcover. □

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